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The Thermodynamic Limit for Long-Range Random Systems

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Long-range spin systems with random interactions are considered. A simple argument is presented showing that the thermodynamic limit of the free energy exists and depends neither on the specific random configuration nor on the sample shape, provided there is no external field. The argument is valid for both classical and quantum spin systems, and can be applied to (a) spins randomly distributed on a lattice and interacting via dipolar interactions; and (b) spin systems with potentials of the form $J(x_1, x_2)/|x_1 - x_2|^{\alpha d}$, where the $J(x_1, x_2)$ are independent random variables with mean zero, d is the dimension, and $\alpha > 1/2$. The key to the proof is a (multidimensional) subadditive ergodic theorem. As a corollary we show that, for random ferromagnets, the correlation length is a nonrandom quantity.

KEY WORDS: Subadditive ergodic theorem; random interactions; long-range interactions; dipolar coupling; free energy; correlation length.

1. INTRODUCTION

Notwithstanding the randomness, the outcomes of most experiments done on random systems, including spin glasses, do not depend on the specific sample and are *reproducible*. In a typical experiment one is concerned with systems in (or near to) equilibrium and measures a thermodynamic observable such as the freezing temperature of a spin glass, the magnetization, susceptibility, or specific heat. It then turns out that all samples whose preparation and external parameters (magnetic field, temperature, etc.) are identical give the same experimental outcomes. That is, one can simply take a *specific* sample and need not average over an ensemble of them.

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Consequently, a theoretical model not only has to give sensible answers for the averages, but it also has to be such that the thermodynamic observables are "self-averaging," i.e., that they converge *with probability one* to a nonrandom answer as the system becomes macroscopic. We note, however, that local quantities such as correlation functions may show a considerable dispersion (scatter) and that probability-one behavior is to be expected for global quantities only. In this paper we consider long-range lattice models, with particular emphasis on their free energy, and take the coupling constants as quenched independent random variables.

Let us denote a finite domain in $\mathbb{Z}^d$ by Λ, the number of sites in Λ by $|\Lambda|$, and the free energy of Λ by $F(\Lambda)$. For short-range random systems it has been shown[1] that $|\Lambda|^{-1}F(\Lambda)$ converges with probability one to a nonrandom limit as $\Lambda \to \infty$ (in the sense of van Hove). The main ingredient of the proof is the observation that the free energy $F(\Lambda)$ of a large system Λ composed of smaller subsystems Λ_i equals the sum of the free energies $F(\Lambda_i)$ of the components, plus a term due to the interactions between different components,

$$F\left(\bigcup_{i=1}^{k} \Lambda_i\right) = \sum_{i=1}^{k} F(\Lambda_i) + B_\Lambda, \qquad \Lambda_i \cap \Lambda_j = \emptyset \quad \text{if} \quad i \neq j \qquad (1.1)$$

If the interactions are short-range, B_Λ represents a small fraction of the bulk free energy $F(\Lambda)$, roughly proportional to the ratio of surface to volume for the components. Thus the free energy of short-range random systems is additive as the size of the system increases to infinity and, hence, allows for application of the well-known additive ergodic theorem,[2] which is a mainstay for the study of disordered materials.

The above argument plainly does not work when the interactions have too long a range (are not summable). If, however, the Hamiltonian is quadratic in the spin operators, the free energy $F(\Lambda)$ is *sub*additive in Λ, i.e.,

$$F\left(\bigcup_{i=1}^{k} \Lambda_i\right) \leqslant \sum_{i=1}^{k} F(\Lambda_i), \qquad \Lambda_i \cap \Lambda_j = \emptyset \quad \text{if} \quad i \neq j \qquad (1.2)$$

This allows us to use the subadditive ergodic theorem of Akcoglu and Krengel.[3] The theorem implies that if (a) $F(\Lambda)$ is subadditive in Λ and (b) the *average* free energy exists as $\Lambda \to \infty$ (in the sense of Fisher), then the thermodynamic limit of $|\Lambda|^{-1}F(\Lambda)$ exists with probability one and is nonrandom. We give sufficient conditions on the coupling constants so as to ensure requirement (b).

For long-range Ising systems the existence of the free energy with probability one has previously been proved by Khanin and Sinai,[4] who also announced the same result for a wider class of *classical* lattice systems. The present approach is much simpler, and applies to both classical and quantum models, whether Ising, Heisenberg, or n-component. It also shows the power of the subadditive ergodic theorem in this type of problem.

In Section 2 we treat the Ising case. We turn to more general classical and quantum models in Section 3, where one can also find a proof of the fact that in random *ferro*magnets the correlation length is a nonrandom quantity. We summarize our results in Section 4.

2. ISING SYSTEMS

We consider a random Ising pair interaction with Hamiltonian

$$H_\Lambda(\{J\}) = -\frac{1}{2} \sum_{i,j\in\Lambda} J(i,j)\Phi(i-j)S(i)S(j) \tag{2.1}$$

Here the $\Phi(i-j)$ determine the range of the interaction and the $J(i,j)$ are *independent* random variables whose distribution only depends on $(i-j)$ and satisfies a uniformity condition; cf. Eq. (2.6b). We always use $\langle \cdots \rangle$ to denote averaging with respect to the random configuration $\{J\}$.

For a particular configuration $\{J\}$ of the random variables the corresponding free energy per site in the thermodynamic limit is given by

$$f(\{J\}) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} F(\Lambda;\{J\}) = -kT \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \ln Z_\Lambda(\{J\}) \tag{2.2}$$

where

$$Z_\Lambda(\{J\}) = \operatorname{Tr}\exp - \beta H_\Lambda(\{J\}) \tag{2.3}$$

is the partition function for the Hamiltonian $H_\Lambda(\{J\})$ and $\beta = 1/kT$ is the inverse temperature, which we put equal to one throughout what follows. For Ising spins $[S(i) = \pm 1]$ the trace means a sum over all $2^{|\Lambda|}$ spin configurations.

As to the existence of the limit in (2.2) we discern two cases.

Case I (Short-range interactions[1]): If

$$\sum_{j\in\mathbb{Z}^d} |\Phi(j)| < \infty \tag{2.4}$$

i.e., Φ in $l^1(\mathbb{Z}^d)$, then the limit (2.2) exists in the sense of van Hove[5] with probability one and is independent of the boundary conditions. Moreover, the stronger notion of thermodynamic convergence holds,[1c] which partly justifies the use of the replica method. The $J(i,j)$ are only required to have a uniformly bounded finite first moment. The proof of (2.2) exploits relation (1.1) and the additive ergodic theorem (or the strong law of large numbers) combined with standard arguments for the deterministic, translation invariant case.[5] For nonrandom, translation invariant spin systems the l^1-condition (2.4) appears necessary. In three dimensions the dipole-dipole interactions are at the borderline: they are not l^1, and the free energy exists only in the Fisher, not in the van Hove, sense.[6]

Case II (Long-range interactions[4]): If

$$\sum_{j \in \mathbb{Z}^d} |\Phi(j)|^2 < \infty \tag{2.5}$$

i.e., Φ in $l^2(\mathbb{Z}^d) \supseteq l^1(\mathbb{Z}^d)$, then the limit (2.2) exists in the sense of Fisher[5] with probability one and is independent of the boundary conditions. The $J(i, j)$ are required to satisfy a moment condition,[4]

$$\langle J(i, j) \rangle = 0 \tag{2.6a}$$

$$|\langle J^n(i, j) \rangle| \leqslant n! c^{n-2} \langle J^2(i, j) \rangle \tag{2.6b}$$

where the second moments $\langle J^2(i, j) \rangle$ are uniformly bounded. Note that all moments must exist. To simplify the notation we choose c in such a way that

$$|\langle J^n(i, j) \rangle| \leqslant n! c^n \tag{2.6c}$$

The proof in Ref. 4 requires the theory of large deviations, is quite complicated, and only works for discrete (Ising) spins. A much weaker result for a restricted class of probability distributions has been obtained by Goulart Rosa,[7] who proves the existence of the *average* free energy, $\lim_{\Lambda \to \infty} |\Lambda|^{-1} \langle F(\Lambda; \{J\}) \rangle$, with free boundary conditions. The limit holds again in the Fisher sense.[2]

In this paper we treat Case II. Except for the fact that we have to restrict ourselves to free boundary conditions, we recover all the results of Ref. 4, but in a much simpler way that also reveals the physics behind the l^2-condition (2.5). Our main tool is a subadditive ergodic theorem of Akcoglu and Krengel (Ref. 3, Theorem 2.7):

Theorem. Let F be a stochastic process on $\mathbb{Z}^d$ such that

(a) $$F\left(\bigcup_{i=1}^{k} \Lambda_i ; \{J\} \right) \leqslant \sum_{i=1}^{k} F(\Lambda_i ; \{J\}), \qquad \Lambda_i \cap \Lambda_j = \emptyset \qquad \text{if} \quad i \neq j$$

(subadditivity);

(b) $$|\Lambda|^{-1} \langle F(\Lambda; \{J\}) \rangle \geqslant C,$$

[2] The claim of van Hove convergence seems to be incorrect. The fact that subadditivity of a function does not suffice for van Hove convergence has been known for quite a long time.[8,9,18] The difference between van Hove and Fisher convergence is, roughly, that for van Hove convergence the surface to volume ratio has to approach zero whereas for Fisher convergence the Λ's have to increase at about the same rate in all directions.

for some constant C independent of Λ. Then $\lim_{\Lambda\to\infty}|\Lambda|^{-1}F(\Lambda;\{J\})$ exists in the sense of Fisher with probability one (for almost every $\{J\}$).

We apply this theorem to the Hamiltonian (2.1) and prove the subadditivity of the corresponding free energy. This was first done by Griffiths,[6] but to make the paper reasonably self-contained we will sketch a slightly different argument leading to (1.2). The Hamiltonian (2.1) is quadratic in the spin operators of Λ and, thus, spin-flip invariant. Let $\rho_{\Lambda_i} = C^{\text{st}}\exp[-\beta H(\Lambda_i;\{J\})]$ be the Gibbs state for Λ_i with free boundary conditions and suppose Λ is the disjoint union of a (finite) number of Λ_i's. By the variational principle [Ref. 5b, pp. 46–47]

$$F\left(\bigcup_{i=1}^{k}\Lambda_i\right)=\min_{\rho}\left\{-S\left(\bigcup_{i=1}^{k}\Lambda_i\,;\rho\right)+\rho\left(H\left(\bigcup_{i=1}^{k}\Lambda_i\right)\right)\right\} \tag{2.7}$$

Only the dependence upon Λ has been made explicit. The last term is the expectation with respect to ρ of the energy, and the entropy functional is given by $S(\Lambda;\rho)=-\operatorname{Tr}\rho\ln\rho$, where the trace is restricted to Λ. The left side of (2.7) is certainly less than the right side if we stick to $\rho=\bigotimes_{i=1}^{k}\rho_{\Lambda_i}$. Then $[\ln(xy)=\ln x+\ln y]$

$$S\left(\bigcup_{i=1}^{k}\Lambda_i\,;\bigotimes_{i=1}^{k}\rho_{\Lambda_i}\right)=\sum_{i=1}^{k}S(\Lambda_i\,;\rho_{\Lambda_i}) \tag{2.8}$$

and

$$\rho\left(H\left(\bigcup_{i=1}^{k}\Lambda_i\right)\right)=\sum_{i=1}^{k}\rho_{\Lambda_i}(H(\Lambda_i))+\text{mixed terms} \tag{2.9}$$

We now come to the heart of the argument. Consider a typical "mixed term": $\rho_{\Lambda_i}\otimes\rho_{\Lambda_j}(S(k)S(l))=\rho_{\Lambda_i}(S(k))\rho_{\Lambda_j}(S(l))$ for k in Λ_i and l in Λ_j with $\Lambda_i\cap\Lambda_j=\varnothing$. Since finite-volume Gibbs states still have *all* the symmetries of the Hamiltonian,

$$\rho_{\Lambda_i}(S(k))=\rho_{\Lambda_j}(S(l))=0 \tag{2.10}$$

and we are left with the homogeneous terms only. Thus

$$F\left(\bigcup_{i=1}^{k}\Lambda_i\right)\leqslant\sum_{i=1}^{k}F(\Lambda_i),\qquad \Lambda_i\cap\Lambda_j=\varnothing\quad\text{if}\quad i\neq j \tag{2.11}$$

which is condition (a) of the subadditive ergodic theorem.

Before proceeding we note that this argument may be readily generalized to classical and quantum Heisenberg and more general, n-component models. In the classical case one performs the canonical transformation $S(i)\to -S(i)$ for all i, and in the quantum case one may use time-reversal; cf. Ref. 6, p. 658. [The time-reversal operator Θ is an antiunitary operator with the property $\Theta S \Theta^{-1} = -S$; moreover, it leaves the trace invariant and commutes with the Hamiltonian (2.1) since $H(\Lambda; \{J\})$ contains only terms quadratic in the spin operators of Λ.]

We now have to exhibit a uniform lower bound for the *average* free energy so as to fulfill condition (b) of the subadditive ergodic theorem. For simplicity of notation we take all the $J(i, j)$ as identically distributed. In view of condition (2.6) this is not a serious restriction. We split up the Hamiltonian (2.1) into a finite-range part $H_F(\Lambda; \{J\})$ containing all terms with $|i-j| \leqslant R$ and a long-range part $H_L(\Lambda; \{J\})$ containing the remaining terms. R is chosen in such a way that $\sum_{|j|>R} |\Phi(j)|^2$ is small enough; this will be specified later. We then get, using a self-evident notation and taking advantage of the concavity of the free energy [cf. Eq. (2.2) and Refs. 5a §2.5, 5b Lemma I.3.3 or, in the classical case, Cauchy–Schwarz],

$$F(\Lambda; H(\Lambda; \{J\})) \geqslant \tfrac{1}{2} F(\Lambda; 2H_F(\Lambda; \{J\})) + \tfrac{1}{2} F(\Lambda; 2H_L(\Lambda; \{J\})) \tag{2.12}$$

Because R has been chosen independently of Λ, the (uniform) lower bound for the H_F-part can be derived by standard methods[1] à la case I, and we are left with H_L.

The moment condition (2.6c) implies that for t sufficiently small the characteristic function $\langle \exp[tJ(i, j)] \rangle$ is analytic in t and, hence, has an absolutely convergent cumulant expansion,[10] which by (2.6a) starts with the *second* cumulant,

$$\langle \exp[tJ(i, j)] \rangle = \exp\left[\sum_{n=2}^{\infty} \langle J^n(i, j) \rangle_c t^n / n! \right] \tag{2.13}$$

In the estimate below this observation is basic. Now, by Jensen's inequality,

$$|\Lambda|^{-1} \langle \ln Z_\Lambda(2H_L\{J\}) \rangle \leqslant |\Lambda|^{-1} \ln \langle Z_\Lambda(2H_L\{J\}) \rangle \tag{2.14}$$

and the right-hand side may be rewritten

$$\begin{aligned}
&|\Lambda|^{-1} \ln \langle Z_\Lambda(2H_L\{J\}) \rangle \\
&= |\Lambda|^{-1} \ln \left\langle \operatorname{Tr} \prod_{\substack{i, j \in \Lambda \\ |i-j|>R}} \exp[J(i, j)\Phi(i-j)S(i)S(j)] \right\rangle \\
&= |\Lambda|^{-1} \ln \left\{ \operatorname{Tr} \prod_{\substack{i, j \in \Lambda \\ |i-j|>R}} \langle \exp[J(i, j)\Phi(i-j)S(i)S(j)] \rangle \right\} \\
&= |\Lambda|^{-1} \ln \left\{ \operatorname{Tr} \prod_{\substack{i, j \in \Lambda \\ |i-j|>R}} \exp\left[\sum_{n=2}^{\infty} \frac{1}{n!} \langle J^n(i, j) \rangle_c (\Phi(i-j)S(i)S(j))^n \right] \right\}
\end{aligned} \tag{2.15}$$

The first equality in (2.15) is nothing but the defintion of Z_Λ with the restriction $|i-j|>R$, for the second equality we used the independence of the $J(i,j)$, and for the last we took advantage of (2.13). Since $|S(i)|=1$, expression (2.15) is majorized by

$$\ln 2+|\Lambda|^{-1}\sum_{n=1}^{\infty}\frac{1}{n!}|\langle J^n(i,j)\rangle_c|\sum_{\substack{i,j\in\Lambda\\|i-j|>R}}|\Phi(i-j)|^n$$

$$\leqslant \ln 2+\sum_{n=\ln 2}^{\infty}\frac{1}{n!}|\langle J^n(i,j)\rangle_c|\left[\sum_{|j|>R}|\Phi(j)|^n\right] \tag{2.16}$$

We finish the argument by noting that (2.5) holds and for $n\geqslant 2$

$$\sum_{|j|>R}|\Phi(j)|^n\leqslant\left(\sum_{|j|>R}|\Phi(j)|^2\right)^{n/2} \tag{2.17}$$

so that, if in (2.13)

$$t^2:=\sum_{|j|>R}|\Phi(j)|^2 \tag{2.18}$$

is small enough, i.e., R is large enough, the series in the right-hand side of (2.16) converges. An upper bound for the partition function gives a lower bound for the free energy and, thus, we are done. In the appendix we show that the thermodynamic limit of the free energy does not depend on the specific random configuration. The shape-independence follows from a slight modification of a standard argument in Ref. 6, Section III.C.

3. EXTENSIONS

The restriction to Ising spins is not essential. The extension of the previous arguments to classical n-component models is immediate: just replace $S(i)S(j)$ in (2.1) by $\sum_{\alpha=1}^{n}S_\alpha(i)S_\alpha(j)$ and notice that all the estimates are still valid, up to a minor modification of (2.16) taking into account the length of the spin.

The result for quantum spins follows from two observations: (a) the subadditivity of the free energy still holds; (b) a classical upper bound for the quantum partition function $Z^Q(S)$ for spin S is obtained[11] by replacing the quantum spin by $(S+1)$ times the classical unit vector,

$$Z^Q(S)\leqslant Z^C(S+1) \tag{3.1}$$

The classical partition function $Z^C(S+1)$, where the spins are vectors on a three-dimensional sphere with radius $(S+1)$ and the usual measure, is easy to handle. Since an upper bound for the partition function gives a lower bound for the free energy, the result is established.

A second application of these arguments is provided by dipoles with random positions on a lattice, whose Hamiltonian is given by

$$H(\Lambda; \{\xi\}) = \frac{1}{2} \sum_{i \neq j} \xi_i \xi_j \sum_{\alpha, \beta} A_{ij}^{\alpha\beta} S_\alpha(i) S_\beta(j) \tag{3.2}$$

where the ξ_i are either zero or one with probability p or $(1-p)$ and the $A_{ij}^{\alpha\beta}$ denote the well-known dipole–dipole coupling constants. The ξ_i are independent, identically distributed random variables which indicate whether a lattice site is occupied or not. We assume there is no external field. In this case also the subadditivity is readily established, and a lower bound for the free energy follows from the inequality[6,12]

$$H(\Lambda; \{\xi\}) \geqslant -C|\Lambda| \tag{3.3}$$

for some constant C independent of Λ and $\{\xi\}$.

Yet another application of the subadditive ergodic theorem (in this case the original one of Kingman[13] suffices) provides us with the existence of the nonrandom correlation length for a class of random Ising *ferro*magnets such as those studied by Fisch.[14] The Hamiltonian reads

$$H(\Lambda; \{J\}) = -\sum_{i, j \in \Lambda} J(i, j) S(i) S(j) \tag{3.4}$$

The $J(i, j)$, which assume positive values, are independent random variables whose distribution only depends on $(i-j)$.

Pick a random configuration $\{J\}$. Since the interaction is ferromagnetic, a Gibbs state μ_β on $\mathbb{Z}^d$ may be obtained as the infinite volume limit of $\mu_{\beta,\Lambda}$ with ferromagnetic boundary conditions. Let

$$\mu_\beta^T(S(i)S(j)) = \mu_\beta(S(i)S(j)) - \mu_\beta(S(i))\,\mu_\beta(S(j)) \geqslant 0 \tag{3.5}$$

be the truncated pair correlation function, which also depends on $\{J\}$. Inequalities of Griffiths and Graham[15] imply

$$\mu_\beta^T(S(j)S(l)) \geqslant \mu_\beta^T(S(j)S(k))\,\mu_\beta^T(S(k)S(l)) \tag{3.6}$$

Moreover, $-\ln \mu_\beta^T(S(i)S(j)) > 0$, whatever j. Suppose i, j, and k in (3.6) are on a straight line in a *fixed* direction $\hat{\mathbf{n}}$. Then $-\ln \mu_\beta^T(S(i)S(j))$ determines a subadditive stochastic process on the line and, by the subadditive ergodic theorem, the limit

$$\lim_{j \to \infty} -\frac{1}{|j|} \ln \mu_\beta^T(S(0)S(j)) = \frac{1}{\xi(\hat{\mathbf{n}})} \tag{3.7}$$

exists with probability one and does *not* depend on $\{J\}$; cf. the Appendix.

Because the symmetry of the lattice is only discrete, a directional dependence of $\xi(\hat{\mathbf{n}})$ upon $\hat{\mathbf{n}}$ cannot be excluded, as is exemplified by the translation invariant nearest-neighbor Ising ferromagnet.[16] We expect a *finite* correlation length $\xi(\hat{\mathbf{n}})$, if the interactions are finite-range or exponentially decaying and the temperature is either high or low, or with an external magnetic field [in that case (3.6) also holds]. See for example Ref. 17. Summarizing: Though the correlation functions of a random ferromagnet may show a considerable dispersion, their asymptotic behavior is almost surely nonrandom.

4. CONCLUSION

We have shown that subadditive ergodic theorems are quite helpful in analyzing the behavior of random systems. In particular, for long-range random spin systems optimal conditions on the range of the interaction and the distribution of the coupling constants show up naturally, and proofs run in a rather direct and simple way. Moreover, we are able to elucidate the physical reason behind the fact that potentials of the form $J(x_1, x_2)/|x_1 - x_2|^{\alpha d}$, where the $J(x_1, x_2)$ are independent random variables *with mean zero*, give rise to a well-defined free energy only if $\alpha > 1/2$. Physically it is plausible that a random variable with mean zero effectively decreases the range of a potential which behaves at infinity like $|x_1 - x_2|^{-\alpha d}$. But why should α exceed precisely $1/2$? This becomes clear if we look at our lower bound for the free energy as it evolves from condition (b) of the subadditive ergodic theorem (Section 2). The lower bound is obtained via the cumulant expansion (2.13) of the characteristic function $\langle \exp[tJ(i,j)] \rangle$. If $\langle J(i,j) \rangle = 0$, the first cumulant vanishes and the series starts with the *second* cumulant, which is the variance of $J(i,j)$ and nonzero whenever the probability distribution of $J(i,j)$ is nontrivial. Studying the transition (2.14)–(2.16) we see that we get a series with lowest-order term

$$\sum_{\substack{j \in \mathbb{Z}^d \\ |j| > R}} |j|^{-2\alpha d} \tag{4.1}$$

which is finite only if $2\alpha d > d$, i.e., $\alpha > 1/2$. If $\langle J(i,j) \rangle \neq 0$, we have to resort to the l^1-condition (2.4), which implies $\alpha > 1$. We note, however, that in both cases thermodynamic observables give rise to *non*random and, thus, reproducible answers as the size of the system becomes macroscopic.

Surprisingly, the dipolar interaction is much easier to deal with. Apparently a physically sensible potential gives rise to a straightforward mathematical treatment once one uses the appropriate formalism—here the subadditive ergodic theorem.

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APPENDIX

Throughout what follows we fix a sequence of Λ's which tend to $\mathbb{Z}^d(\Lambda \to \infty)$ in the sense of Fisher. By the subadditive ergodic theorem (Sections 2 and 3) the limit

$$f(\{J\}) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} F(\Lambda; \{J\}) \tag{A.1}$$

exists with probability one, but f may still depend on the specific random configuration $\{J\}$. We now show that this is not the case.

The random variables $J(i, j)$ are to be associated with a measure space $(\Omega, \mathscr{F}, \mu)$ where the $J(i, j)$ are integrable with respect to μ. The free energy $F(\Lambda; \{J\})$ is a set function on $\mathbb{Z}^d$ which maps each finite domain Λ onto the stochastic variable $F(\Lambda; \{J\})$; cf. Eqs. (2.1)–(2.3). Let a be a vector in $\mathbb{Z}^d$ and define τ_a to be the shift in Ω which shifts each configuration $\{J\}$ by $-a$, i.e.,

$$(\tau_a J)(i, j) = J(i - a, j - a) \tag{A.2}$$

Since τ_a is one-to-one and measure preserving, it induces an isometry U_a in $L^1_\mu(\Omega)$ by

$$L^1_\mu(\Omega) \ni f \to (U_a f)(\{J\}) = f(\tau_a\{J\}) \tag{A.3}$$

By (2.1)–(2.3)

$$[U_a F(\Lambda; \cdot)](\{J\}) = F(\Lambda + a; \{J\}) \tag{A.4}$$

Since the $J(i, j)$ are *independent* random variables whose distribution only depends on $(i - j)$, the transformation τ_a is ergodic and, hence (Walters[2] p. 23),

$$U_a f = f \Rightarrow f = C\mathbb{1} \qquad \text{(a.e. } [\mu]) \tag{A.5}$$

We show that f in (A.1) is translation invariant. In doing so we take advantage of (A.4).

Label the Λ's in such a way that $\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots$. Since $\Lambda_n \to \infty$ in the sense of Fisher, so do $\{\Lambda_n + a\}$ and the "intertwining" sequence $\{\Lambda'_n\}$ defined by $\Lambda'_{2n} = \Lambda_n$, $\Lambda'_{2n+1} = \Lambda_n + a$. Hence $|\Lambda'_n|^{-1} F(\Lambda'_n; \{J\})$ converges with probability one to a finite limit as $n \to \infty$ and so do the subsequences with n even and odd, so that by (A.4), for $[\mu]$-almost every $\{J\}$, $f(\{J\}) = (U_a f)(\{J\})$, and the proof is complete.

Now consider the average $|\Lambda|^{-1}\langle F(\Lambda; \{J\})\rangle$. By the subadditivity this function is decreasing in Λ and by condition (b) of the subadditive ergodic theorem (Section 2) it has a *finite* limit as $\Lambda \to \infty$. Moreover, we have for almost every $\{J\}$,

$$f(\{J\}) = \lim_{\Lambda\to\infty} |\Lambda|^{-1}\langle F(\Lambda, \{J\})\rangle = \inf_{\Lambda} |\Lambda|^{-1}\langle F(\Lambda; \{J\})\rangle \quad \text{(A.6)}$$

This justifies the customary procedure of calculating the free energy per site for a specific sample.

We finish this appendix by indicating another proof of the translational invariance of f. To simplify the argument we take the one-dimensional subadditive process (3.7) of Section 3, put $f(\{J\}) = \lim_{n\to\infty} n^{-1}F([0,n]; \{J\})$, suppress the dependence upon $\{J\}$ in F, and note

$$F([0, n+1]) \leqslant F(\{0\}) + F([1, n+1]) \quad \text{(A.7)}$$

so that

$$f(\{J\}) = \lim_{n\to\infty} n^{-1}F([0, n+1]) \leqslant \lim_{n\to\infty} n^{-1}F([1, n+1]) = (f \circ \tau_1)(\{J\}) \quad \text{(A.8)}$$

On the other hand, because τ_1 is measure preserving,

$$\int_\Omega \{f \circ \tau_1 - f\}\, d\mu = 0 \quad \text{(A.9)}$$

Thus the integrand, which is nonnegative by (A.8), vanishes $[\mu]$-almost everywhere, and the translational invariance of f is established.

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